Some Properties of Parametrized Fixed Points on O-Categories

And Applications to Session Types

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Session-Typing Communication

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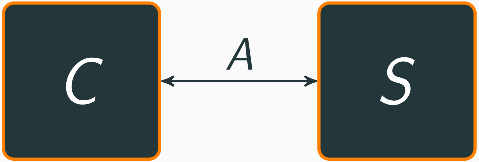

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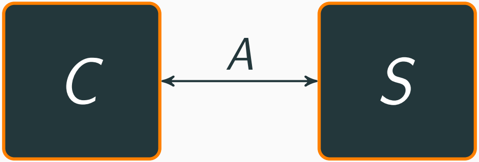
Session-Typing Communication



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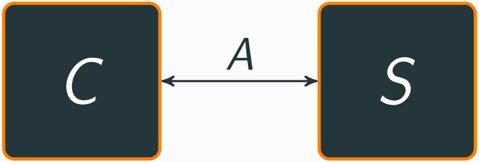

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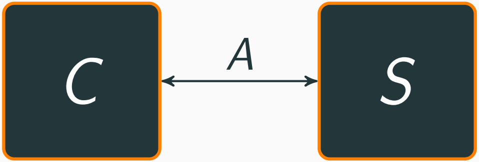

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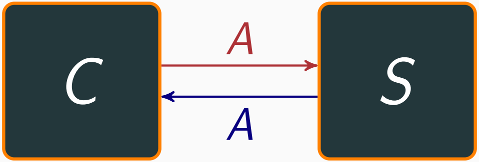

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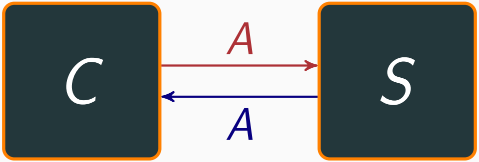

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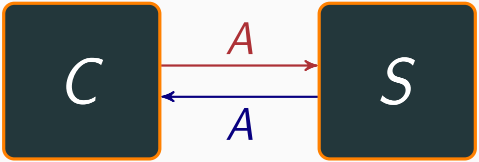

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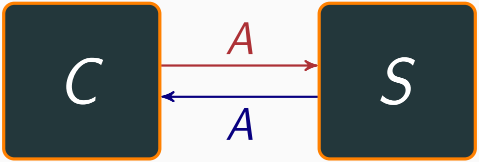

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Want an embedding $[\![A]\!] \to [\![A]\!] \times [\![A]\!]$.

Open Session-Types

Generalize $[\![A]\!] \to [\![A]\!] \times [\![A]\!]$ to open session types $\Xi \vdash A$:

$$\mathbf{DCPO}^{\Xi} = \prod_{\alpha \in \Xi} \mathbf{DCPO}$$

$$[\![\Xi \vdash A]\!] \, , [\![\Xi \vdash A]\!], [\![\Xi \vdash A]\!] : \mathbf{DCPO}^{\Xi} \xrightarrow{\text{l.c.}} \mathbf{DCPO}$$

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The embedding becomes a natural transformation:

$$(\!|\Xi \vdash A|\!) : [\![\Xi \vdash A]\!] \Rightarrow [\![\Xi \vdash A]\!] \times [\![\Xi \vdash A]\!]$$

where each component of $(\!|\Xi \vdash A|\!)$ is an embedding.

Recursive Session-Types

Recursive session types are formed by the following rule:

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Should respect unfolding, e.g.,

$$[\![\Xi \vdash \mathtt{rec}(\alpha.A)]\!] \cong [\![\Xi \vdash [\mathtt{rec}(\alpha.A)/\alpha]A]\!] = [\![\Xi, \alpha \vdash A]\!]\,(-, [\![\Xi \vdash \mathtt{rec}(\alpha.A)]\!])$$

Parametrized Families of Fixed Points

Given $F : \boldsymbol{DCPO}^{\Xi,\alpha} \to \boldsymbol{DCPO}$, we can find a parametrized family of fixed points $F^\dagger : \boldsymbol{DCPO}^{\Xi} \to \boldsymbol{DCPO}$ by taking:

$$F^\dagger D = \mathrm{FIX}(F(D, -)).$$

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Example: if $F = [\![\Xi, \alpha \vdash A]\!]$ and $[\![\Xi \vdash \mathtt{rec}(\alpha.A)]\!] = F^\dagger$, then

$$[\![\Xi \vdash \mathtt{rec}(\alpha.A)]\!] \cong [\![\Xi, \alpha \vdash A]\!]\,(-, [\![\Xi \vdash \mathtt{rec}(\alpha.A)]\!]).$$

Interpreting Recursive Session Types

Is this $(\cdot)^\dagger$ functorial? Does it preserve embeddings?

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Wrong: Take $(\!|\, \Xi \vdash \texttt{rec}(\alpha.A) \,|\!)$ to be

$$(\!|\, \Xi, \alpha \vdash A \,|\!)^\dagger : [\![\Xi, \alpha \vdash A]\!]^\dagger \Rightarrow ([\![\Xi, \alpha \vdash A]\!] \times [\![\Xi, \alpha \vdash A]\!])^\dagger \,.$$

Problem: What are $[\![\Xi \vdash \texttt{rec}(\alpha.A)]\!]$ and $[\![\Xi \vdash \texttt{rec}(\alpha.A)]\!]$?
In general, $(F \times G)^\dagger \not\cong F^\dagger \times G^\dagger$.

This Talk

We give a parametrized fixed-point operator $(\cdot)^\dagger$ on $\boldsymbol{O}$-categories suitable for interpreting recursive session types.

It is locally continuous and satisfies the Conway identities.

O-Categories and Local Continuity

Definition

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Definition

An **embedding-projection pair** (**e-p-pair**) is a pair of morphisms $e : A \to B$ and $p : B \to A$ such that $p \circ e = \mathrm{id}_A$ and $e \circ p \sqsubseteq \mathrm{id}_B$.

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Output: Object $\mathsf{FIX}(F)$ such that $F(\mathsf{FIX}(F)) \cong \mathsf{FIX}(F)$

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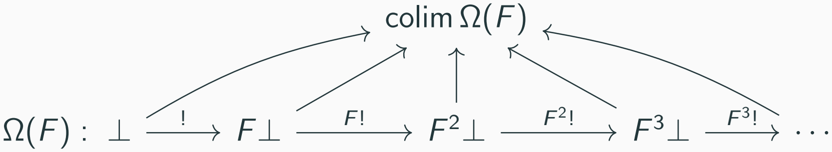

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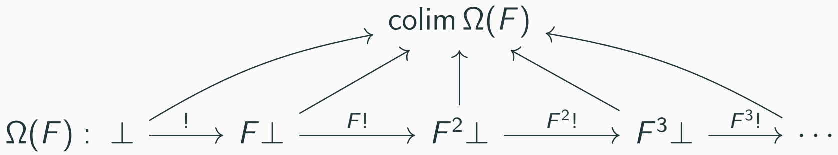

Define: $\mathsf{FIX}(F) = \operatorname{colim} \Omega(F)$.

Generalizing Links

Let $\mathbf{Links}_{\mathbf{K}}$ be:

Objects: (C, c, F) where $F : \mathbf{K} \xrightarrow{\text{l.c.}} \mathbf{K}$ and $c : C \to FC$ is an embedding;

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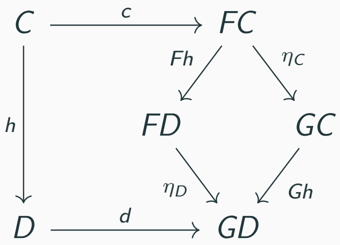

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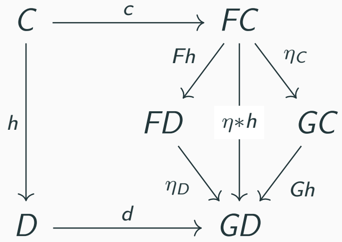

Write $\eta * h$ for $\eta_D \circ Fh = Gh \circ \eta_C$.

Generalizing Chains

Define $\Omega : \mathbf{Links}_{\boldsymbol{K}} \to [\omega \to K]$ by

$$\Omega(C, c, F) : \quad C \xrightarrow{c} FC \xrightarrow{Fc} F^2C \xrightarrow{F^2c} F^3C \xrightarrow{F^3c} \cdots$$

Generalizing Chains

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\Downarrow \Omega(h,\eta) & \downarrow h & & \downarrow \eta * h & & \downarrow \eta^{(2)} * h & & \downarrow \eta^{(3)} * h & & \\
\Omega(D, d, G): & D & \xrightarrow{d} & GD & \xrightarrow{Gd} & G^2D & \xrightarrow{G^2d} & G^3D & \xrightarrow{G^3d} & \cdots
\end{array}$$

where $\eta^{(n)} = \eta * \cdots * \eta : F^n \Rightarrow G^n$ for $\eta : F \Rightarrow G$.

General Fixed Points

Proposition. $\mathrm{GFIX} = \mathrm{colim} \circ \Omega : \mathbf{Links}_{\boldsymbol{K}} \xrightarrow{\text{l.c.}} K$ is locally continuous.

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Proposition. $\mathrm{UNF}(C, c, F) = F(\mathrm{GFIX}(C, c, F))$ extends to a functor $\mathbf{Links}_{\boldsymbol{K}} \xrightarrow{\text{l.c.}} \boldsymbol{K}$.

Theorem. There exists a natural isomorphism

$$\mathrm{fold} : \mathrm{UNF} \Rightarrow \mathrm{GFIX},$$

i.e., for all $(h, \eta) : (C, c, F) \to (D, d, G)$,

$$\begin{array}{ccc} F(\mathrm{GFIX}(C, c, F)) & \xrightarrow{\mathrm{fold}_{(C,c,F)}} & \mathrm{GFIX}(C, c, F) \\ {\scriptstyle \mathrm{UNF}(h,\eta)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{GFIX}(h,\eta)} \\ G(\mathrm{GFIX}(D, d, G)) & \xrightarrow{\mathrm{fold}_{(D,d,G)}} & \mathrm{GFIX}(D, d, G). \end{array}$$

Parametrized Fixed Points

Remark. The mapping $F \mapsto (\bot, !, F)$ embeds the category $[K \xrightarrow{\text{l.c.}} K]$ of locally continuous functors on K into $\mathbf{Links}_{\boldsymbol{K}}$.

Parametrized Fixed Points

Remark. The mapping $F \mapsto (\perp, !, F)$ embeds the category $[K \xrightarrow{\text{l.c.}} K]$ of locally continuous functors on K into $\mathbf{Links}_{\boldsymbol{K}}$.

Corollary. The functor $(-)^{\dagger}$ given by

$$[\boldsymbol{D} \times \boldsymbol{K} \xrightarrow{\text{l.c.}} \boldsymbol{K}] \xrightarrow{\Lambda} [\boldsymbol{D} \xrightarrow{\text{l.c.}} [\boldsymbol{K} \xrightarrow{\text{l.c.}} \boldsymbol{K}]] \xrightarrow{[\text{id}_{\boldsymbol{D}} \to \text{GFIX}(\perp,!,-)]} [\boldsymbol{D} \xrightarrow{\text{l.c.}} \boldsymbol{K}]$$

is locally continuous.

Weak Fixed-Point Identity

Theorem. Let $F : \boldsymbol{D} \times \boldsymbol{K} \xrightarrow{\text{l.c.}} \boldsymbol{K}$. There exists a natural isomorphism

$$\mathrm{Fold}^F : F^\dagger \Rightarrow F \circ \langle \mathrm{id}_D, F^\dagger \rangle$$

given by $\mathrm{Fold}^F_D = \mathrm{fold}_{(\perp, !, F(D,-))} : F^\dagger D \to F(D, F^\dagger D)$.
The definition of Fold is also natural in F.

Parameter Identity

Theorem. Let $F, H : \boldsymbol{D} \times \boldsymbol{K} \xrightarrow{\text{l.c.}} \boldsymbol{K}$ and $G, I : \boldsymbol{C} \xrightarrow{\text{l.c.}} \boldsymbol{D}$. Set $F_G = F \circ (G \times \mathrm{id})$ and analogously for H_I. Then

$$F_G^\dagger = F^\dagger \circ G$$
$$\mathrm{Fold}^{F_G} = \mathrm{Fold}^F G.$$

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$$F_G^\dagger = F^\dagger \circ G$$
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If $\phi : F \Rightarrow H$ and $\gamma : G \Rightarrow I$, then

$$(\phi * (\gamma \times \mathrm{id}))^\dagger = \phi^\dagger * \gamma : F_G^\dagger \Rightarrow H_I^\dagger.$$

Parametrized Algebras

Let $F : \boldsymbol{D} \times \boldsymbol{K} \xrightarrow{\text{l.c.}} \boldsymbol{K}$.

Definition. An F-**algebra** is a pair (G, γ) where $G : \boldsymbol{D} \xrightarrow{\text{l.c.}} \boldsymbol{K}$ and $\gamma : F \circ \langle \mathrm{id}, G \rangle \Rightarrow G$.

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Definition. An F-**algebra homomorphism** $(G, \gamma) \to (H, \eta)$ is a $\rho : G \Rightarrow H$ such that

$$\begin{array}{ccc} F \circ \langle \mathrm{id}, G \rangle & \stackrel{\gamma}{\Longrightarrow} & G \\ {\scriptstyle F \circ \langle \mathrm{id}, \rho \rangle} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \rho} \\ F \circ \langle \mathrm{id}, H \rangle & \stackrel{\eta}{\Longrightarrow} & H. \end{array}$$

Canonicity of Parametrized Fixed Points

Theorem. Let $F : \boldsymbol{D} \times \boldsymbol{K} \xrightarrow{\text{l.c.}} \boldsymbol{K}$.

- $(F^\dagger, \mathrm{Fold}^F)$ is the initial F-algebra.
  Given any other F-algebra (G, γ), the unique morphism $\phi : F^\dagger \Rightarrow G$ is a natural family of embeddings.
- $\left(F^\dagger, \left(\mathrm{Fold}^F\right)^{-1}\right)$ is the terminal F-coalgebra.
  Given any other F-coalgebra (G, γ), the unique morphism $\rho : G \Rightarrow F^\dagger$ is a natural family of projections.

Revisiting Recursive Session Types

The interpretation

$$\llparenthesis \Xi \vdash \texttt{rec}(\alpha.A) \rrparenthesis : [\![\Xi \vdash \texttt{rec}(\alpha.A)]\!] \Rightarrow [\![\Xi \vdash \texttt{rec}(\alpha.A)]\!] \times [\![\Xi \vdash \texttt{rec}(\alpha.A)]\!]$$

is given by

$$\langle (\pi_1 \llparenthesis \Xi, \alpha \vdash A \rrparenthesis)^\dagger, (\pi_2 \llparenthesis \Xi, \alpha \vdash A \rrparenthesis)^\dagger \rangle : [\![\Xi, \alpha \vdash A]\!]^\dagger \Rightarrow [\![\Xi, \alpha \vdash A]\!]^\dagger \times [\![\Xi, \alpha \vdash A]\!]^\dagger.$$

It is a natural family of embeddings by the theorem on the previous slide.

Conway Identities

The **Conway identities** are four identities for dagger operations useful for semantic reasoning. They include:

1. the **parameter identity** (**naturality**):
   for all $f : B \times C \to C$ and $g : A \to B$,

$$(f \circ (g \times \mathrm{id}_C))^\dagger = f^\dagger \circ g.$$

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2. the **composition identity** (**parametrized dinaturality**):
   for all $f : P \times A \to B$ and $g : P \times B \to A$,
   $$(g \circ \langle \pi_P^{P\times A}, f\rangle)^\dagger = g \circ \langle \mathrm{id}_P, (f \circ \langle \pi_P^{P\times B}, g\rangle)^\dagger\rangle.$$

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Theorem. The dagger $(\cdot)^\dagger$ satisfies the Conway identities.

Conway Identities: Application

The Conway identities imply:

Corollary (Pairing / Bekič's Identity). Let $F : \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \xrightarrow{\text{l.c.}} \boldsymbol{B}$ and $G : \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \xrightarrow{\text{l.c.}} \boldsymbol{C}$. Set

$$H = \boldsymbol{A} \times \boldsymbol{B} \xrightarrow{\langle \mathrm{id}, G^\dagger \rangle} \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \xrightarrow{F} \boldsymbol{B}.$$

Then

$$\langle F, G \rangle^\dagger = \langle G^\dagger \circ \langle \mathrm{id}_{\boldsymbol{A}}, H^\dagger \rangle, H^\dagger \rangle : \boldsymbol{A} \to \boldsymbol{B} \times \boldsymbol{C}.$$

Conway Identities: Application

The Conway identities imply:

Corollary (Pairing / Bekič's Identity). Let $F : \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \xrightarrow{\text{l.c.}} \boldsymbol{B}$ and $G : \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \xrightarrow{\text{l.c.}} \boldsymbol{C}$. Set

$$H = \boldsymbol{A} \times \boldsymbol{B} \xrightarrow{\langle \mathrm{id}, G^\dagger \rangle} \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \xrightarrow{F} \boldsymbol{B}.$$

Then

$$\langle F, G \rangle^\dagger = \langle G^\dagger \circ \langle \mathrm{id}_{\boldsymbol{A}}, H^\dagger \rangle, H^\dagger \rangle : \boldsymbol{A} \to \boldsymbol{B} \times \boldsymbol{C}.$$

Application. Interpreting and reasoning about mutually recursive session types.

Applications to Session Types

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2. process composition is associative
3. large class of η-like properties

Summary

We gave a parametrized fixed-point operator $(\cdot)^\dagger$ that is:

- **locally continuous**;
- satisfies the **Conway identities**;
- useful for interpreting **recursive session types**.

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