EXPLORATIONS ON THE WALLACE-BOLYAI-GERWIEN THEOREM

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Abstract. In this survey paper, we present a proof of the Wallace-Bolyai-Gerwien theorem, namely, that any two plane polygons of the same area may be decomposed into the same number of pairwise congruent triangles. Several generalisations and closely related theorems will be considered, and an original example will be explored.

1. Introduction

In 1814, Wallace [WL14] posed:

> Is it possible in every case to divide each of two equal but dissimilar rectilinear figures, into the same number of triangles, such, that those which constitute the one figure are respectively identical with those which constitute the other?

That same year, Lowry provided a proof in the affirmative. Perhaps due to its intuitive nature, variants of Wallace's question have been independently posed and answered several times. For example, a positive answer was independently conjectured and proved in 1833 by Gerwien [Ger33a], who also generalised it to spherical polygons [Ger33b]. In 1832, Farkas Bolyai [Bol04, pp. 108ff.] showed any plane polygon could be "transmuted" into a rectangle with an equal area, a step that, as we shall see, is an integral part of Lowry's proof and from which the remainder of the proof readily follows. In 1912, Jackson [Jac12] again independently discovered the result, with a proof identical to Lowry's. Due to their contributions, this theorem is now known as the *Wallace-Bolyai-Gerwien theorem*. Readers interested in a detailed account of this theorem's history are referred to Bartocci [Bar12, pp. 21–39].

Given its frequent appearance in mathematics, it is not surprising that Hilbert was well aware of the theorem. Indeed, it makes an unnamed appearance in his axiomatisation of geometry as "Theorem 30" of *The Foundations of Geometry* [Hil02b]. Moreover, Hilbert's [Hil02a] third problem generalised Wallace's question to the decomposition of polyhedra into tetrahedra.

In section 2, we will revisit the notions of polygons and area and dispatch an unsettling circularity between common notions of area and the above theorem. Then, in section 3, we present Lowry's, Jackson's, and Hartshorne's similar proofs. In section 4, we will apply Lowry's technique to two polygons with equal area to show how they may be dissected into pairwise congruent triangles. Finally, in section 5, we will conclude by considering various generalisations and questions that naturally arise from this theorem, before posing our own open questions.

2. Polygons and Area

We will assume throughout that we are working in a Hilbert plane satisfying Hilbert's parallel axiom and the Archimedean axiom. We recall the statement of the parallel axiom [Hil02b, §5]:

> In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a.

2010 *Mathematics Subject Classification.* Primary 51M04.

The statement of the Archimedian axiom is [Har00, p. 115]:

> Given line segments $\overline{AB}$ and $\overline{CD}$, there is a natural number n such that n copies of $\overline{AB}$ added together will be greater than $\overline{CD}$.

We next recall the standard notion of polygon, using a formulation from Hilbert [Hil02b, p. 9].

Definition 1. A system of segments $\overline{p_1p_2}, \overline{p_2p_3}, \dots, \overline{p_{n-1}p_n}, \overline{p_np_{n+1}}$ is called a *broken line*. If $p_{n+1} = p_1$, then we call this broken line a *polygon*, the segments forming it *sides* or *edges*, and the points p_i its *vertices*. A polygon is called *simple* if its points are distinct, none of them lie in any of the sides forming the polygon, and no two of its edges intersect.

It is intuitively clear that every simple polygon divides the plane into an *interior region* and an *exterior region*. This intuition can be made precise as follows:

Proposition 1 ([Hil02b, Theorem 6]). *Every simple polygon divides the points not lying on its sides into two disjoint regions, an interior and an exterior, such that:*

- *If A is a point of the interior and B a point of the exterior, then any broken line joining them intersects the polygon in at least one point.*
- *If A and A' are two points of the interior, and B and B' are two points of the exterior, then there always exist broken lines joining A to A', and B to B' that do not intersect the polygon.*
- *There exist straight lines in the plane which lie entirely outside of the polygon, but none which lie entirely within it.*

Questions of "area" revolve around this interior region, and have historically involved a certain amount of vagueness. For example, Euclid never explicitly defined his notion of "area", and although in modern times, we treat area as a numeric measurement, this was not the case for him. Rather, he treated it as an undefined notion of equality satisfying the following properties [Har00, p. 196]:

(1) Congruent figures are "equal".
(2) Sums of "equal" figures are "equal".
(3) Differences of "equal" figures are "equal".
(4) Halves of "equal" figures are "equal".
(5) The whole is greater than the part.
(6) If squares are "equal", then their sides are equal.

In 1878, French mathematician Duhamel claimed that two sizes of the same type are equivalent if they are composed of parts which are respectively equal.[1] Using the notion of "decomposition", Hilbert's formalisation of area echoes this, and provides two notions reflecting Euclid's intuitive notion of area. A non-self-intersecting broken line joining two points of a polygon P and lying completely in its interior *decomposes* P into two polygons P_1 and P_2 with disjoint interiors, each of which is a subset of the interior of P; we also say that P is *composed* of P_1 and P_2. For example, a diagonal decomposes a square into two triangles, and a pentagon into a triangle and a quadrilateral. We are now free to define the notions:

Definition 2 (Equal area [Hil02b, p. 58]). Two polygons are said to be of *equal area* when they can be decomposed into a finite number of triangles which are respectively congruent to one another in pairs.

[1] "Deux grandeurs d'espèce quelconque sont dites *équivalentes*, quand elles sont composées de parties respectivement égales, [...]". [Duh78, p. 446]

Definition 3 (Equal content [Hil02b, p. 58])**.** Two polygons are said to be of *equal content* when it is possible, by the addition of other polygons having equal area, to obtain two resulting polygons having equal area.

Although these notions appear in multiple sources, nomenclature unavoidably varies by author. For example, Hartshorne [Har00] calls these notions *equidecomposability* and *equal content*, respectively, and calls a notion of equal area permitting decomposition into arbitrary congruent polygons *equivalence by dissection*; Jackson [Jac12] calls this last notion *congruence by dissection*. Although polygons with equal area clearly have equal content [Har00, Proposition 22.3], these notions are distinct for general Hilbert planes (see, e.g., Hilbert [Hil02b, p. 34] for a construction of two congruent triangles with equal content but unequal area, or Hartshorne [Har00, Example 22.1.3] for a similar construction using parallelograms); they coincide when the Archimedian axiom is satisfied [Har00, 22.1.4, 24.7.3, & 36.7.1].

But herein lies the difficulty with the Wallace-Bolyai-Gerwien theorem. It is true by definition under Hilbert's definition of area, and similarly true under Euclid's implicit but undefined notion of "area", which Hartshorne takes to be "equal content". It is only a careful study of the proof that reveals that the notion of area implicit in the theorem is that of a *numeric measurement*. Indeed, although the German geometric tradition follows Euclid in treating area as a purely geometric construction, the American and English traditions of the era treat area as a *number* [Jac12, p. 384] satisfying at least the following axioms:

(1) The area of a geometric figure is a number.
(2) Congruent figures have the same area.
(3) The area of a figure is the sum of the areas of its parts.
(4) If equal areas are added to equal areas, the wholes are of equal area.
(5) If equal areas are removed from equal areas, the remainders are of equal area.
(6) The halves of equal areas are of equal area.

Here, it is implicit that the notion of "number" is that of a real number. However, since we are working over Hilbert planes, we must be careful in choosing our notion of measurement, and various choices for such a notion exist. For example, Hartshorne [Har00, §23] defines a measure of area function to be a positive function into an ordered group whose restriction to triangles satisfies axioms 2 and 4, and then proceeds to prove various properties, including that such a function exists. For the sake of simplicity, we follow Hilbert's path and, where the base and height of a triangle are defined in the usual way, define:

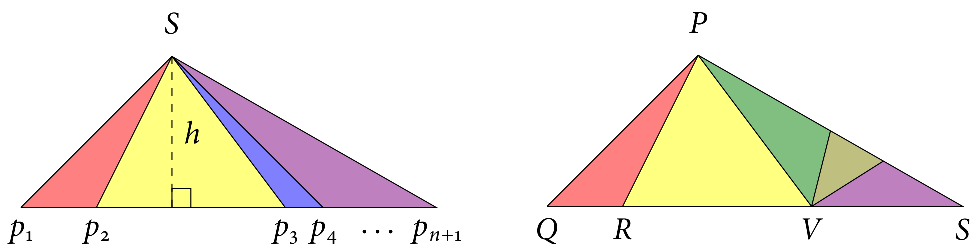
Definition 4. The *measure of area* $\mathrm{F}(\Delta)$ *of the triangle* Δ is given by half of the product of its base and height.

This measure is an element of the field induced by segment arithmetic. It is not difficult to show that this function is well-defined, i.e., independent of the choice of base and altitude; see Hilbert [Hil02b, §20] or Hartshorne [Har00, Lemma 23.3] for details. Hilbert then extends this to:

Definition 5. The *measure of area* $\mathrm{F}(P)$ *of a polygon* P is given by the sum of measures of area of a given finite decomposition of it into triangles.

It is not immediately clear that this function is well-defined, i.e., independent of the choice of triangulation. Fortunately, we have:

Proposition 2 ([Hil02b, Theorem 29; Har00, Lemma 23.5])**.** *The measure of area function for polygons is well-defined.*



(A) Transversal triangulation (B) Transversal subtriangulations

FIGURE 1. Various triangulations

Proof (sketch). Though elementary, the proof is sufficiently long that we can only afford to give a sketch, which we base off of Hartshorne [Har00, pp. 207–210].

We first show that given any triangle T, $\mathrm{F}(T)$ is independent of triangulation, i.e., that for any triangulation T_i of T, we have:

$$\mathrm{F}(T) = \sum_i \mathrm{F}(T_i). \tag{1}$$

The fundamental case is when T is divided into a *transversal triangulation*, that is to say, when the triangulation is determined by segments joining a common vertex S with endpoints $p_1, \dots, p_{n+1}$ on the opposite side, as in Figure 1a. Taking this opposite side b to be the base, we see that all of the triangles $T_i = p_i p_{i+1} S$ in the triangulation have the same altitude h as the triangulated triangle T. Then, due to the distributivity of our field, we have that:

$$F(T) = \frac{1}{2} hb = \frac{1}{2} h \left(\overline{p_1 p_2} + \cdots + \overline{p_n p_{n+1}} \right) = \sum_{i=1}^{n} \frac{1}{2} h \overline{p_i p_{i+1}} = \sum_{i=1}^{n} F(T_i),$$

and so (1) holds when the triangulation is transversal.

The second case consists of a transversal triangulation where we permit the triangles in the triangulation to themselves be transversally triangulated, provided that at least one T's edges has no new vertices and that there are no new vertices in the interior of T. An example of this case is illustrated by Figure 1b, where $\overline{PQ}$ has no new vertices, the triangle VSP is itself transversally triangulated, and there are no new vertices in the interior of the triangle. This case is proven by induction on the number of triangles forming the triangulation. If there are only two, then we fall into the previous case, so suppose there are more than two. Then the side with no new vertices $\overline{PQ}$ belongs to a triangle T_1 of the triangulation, and its third vertex R must lie on one of the other two sides, say $\overline{QS}$. Then the triangle PRS is triangulated as per the hypotheses of this second case and it has one less triangle in its triangulation. Thus, by the induction hypothesis, we have that $\mathrm{F}(PRS) = \sum_{i=2}^{n} T_n$. But by the first case, we also have that $\mathrm{F}(T) = \mathrm{F}(PQR) + \mathrm{F}(PRS) = \mathrm{F}(T_1) + \mathrm{F}(PRS)$, so combining these two results, we get $\mathrm{F}(T) = \sum_{i=1}^{n} T_i$. Thus, when a triangle is triangulated as per the hypotheses of this case, (1) holds as desired.

The general case treats arbitrary triangulation of T, and its proof consists of reducing such triangulations back to the second case. Its proof is omitted due to space constraints.

Having shown that $\mathrm{F}(T)$ is well-defined for triangles T, we now turn to arbitrary polygons P. Suppose we are given given two triangulations T_i and T'_j of a polygon P, then we can straightforwardly adapt the proof of Proposition 3 below. The general idea is to consider the intersections I_{ij} of the T_i and T'_j, themselves triangulated into $I_{ij} = \bigcup_k T_{ijk}$ if I_{ij} is not a triangle. Then by the general case for triangles, we have that each $\mathrm{F}(T_i) = \sum_{j,k} \mathrm{F}(T_{ijk})$ and $\mathrm{F}(T'_j) = \sum_{i,k} \mathrm{F}(T_{ijk})$, and so

$$\sum_i \mathrm{F}(T_i) = \sum_{i,j,k} \mathrm{F}(T_{ijk}) = \sum_j \mathrm{F}(T'_j).$$

We thus conclude the proposition. □

Since F is well-defined, if two polygons have equal area, they clearly have equal measure of area. Having thus laid the stage, we can now interpret the Wallace-Bolyai-Gerwien theorem as the converse of this observation, namely, that if two polygons have equal measure of area, they have equal area. This observation and its converse correspond to Hilbert [Hil02b, Theorem 30], while the converse alone is Hartshorne [Har00, Proposition 23.7].

We take a moment to prove several propositions relating our notions of area. The first relates Jackson's congruence by dissection to our notion of equal area; it will be useful for bringing Jackson's proofs into Hilbert's system, and it appears as a brief remark without proof in Hartshorne [Har00, p. 213].

Proposition 3. *Two polygons are congruent by dissection if and only if they have equal area.*

Proof. Suppose the polygons A and B are congruent by dissection, and are decomposed into respectively congruent pieces $A_1, \dots, A_n$ and $B_1, \dots, B_n$. Then for each i, decompose A_i and B_i into respectively congruent triangles $A_{i1}, \dots, A_{im_i}$ and $B_{i1}, \dots, B_{im_i}$. Then A_i and B_i have equal area, and, considering all A_{ij} and B_{ij}, clearly, so do A and B. The converse is immediate. □

The second states that equality of area is an equivalence relation, and appears in several sources [Hil02b, Theorem 24; Jac12, Theorem I; Har00, Proposition 22.2]. Here, we follow Jackson's proof:

Proposition 4 ([Jac12, Theorem I])**.** *Two polygons each congruent by dissection with a third are congruent by dissection with each other.*

Proof. Assume that the polygons A and B are each congruent by dissection with C. Then A can be decomposed into polygons $A_1, \dots, A_r$ and C into respectively congruent polygons $A'_1, \dots, A'_r$. Similarly, B can be decomposed into polygons $B_1, \dots, B_s$ and C into respectively congruent polygons $B'_1, \dots, B'_s$. Then setting $[A'_i B'_j]$ to be the intersection of A'_i and B'_j, we have that A'_i can can be dissected into $[A'_i B'_1], \dots, [A'_i B'_s]$, and since A_i is congruent to A'_i, it can be dissected into portions respectively congruent to these. Thus, C can be dissected into $\left\{ [A'_i B'_j] \right\}$ $(1 \le i \le r, 1 \le j \le s)$, and A can be dissected into respectively congruent $\left\{ [A_i B_j] \right\}$. But B can in the same way be dissected into parts respectively congruent to the $\left\{ [A'_i B'_j] \right\}$ forming C, and so by transitivity of congruence of polygons, A and B are congruent by dissection. □

Corollary 1. *Two polygons each having equal area to a third themselves have area equal. Thus, equality of area is an equivalence relation.*

Proof. The first claim is immediate by Propositions 3 and 4. As for equivalence, reflexivity and symmetry are obvious, while symmetry and the proposition give transitivity. □

3. Lowry's, Jackson's, and Hartshorne's Proofs

We consider the proofs by Lowry, Jackson, and Hartshorne, all of which follow the same general structure. We begin with a short lemma:

Lemma 1 ([WL14, Prob. I])**.** *To decompose a given triangle into parts which shall form a rectangle. (Figure 2)*

Proof. Let $\overline{AB}$ be the longest side of a triangle ABE, and let H and G be the midpoints of $\overline{AE}$ and $\overline{BE}$ respectively. Then, by Euclid [Euc08b, VI.2], the lines HG and AB are parallel. Extend the segment $\overline{HG}$ so that it intersects the perpendiculars at A and B of $\overline{AB}$ in C and D, respectively. Now, let $\overline{FE}$ be perpendicular to $\overline{HG}$ with F on $\overline{HG}$. Then by Euclid [Euc08a, I.15], the angle $\angle FGE$ is congruent to the angle $\angle BGD$. Thus, by Euclid [Euc08a, I.26], the triangles FEG and GDB are congruent; symmetrically, ACH and HFE are congruent. But then, it is easy to see that $ABDC$ and AEB have equal area. □

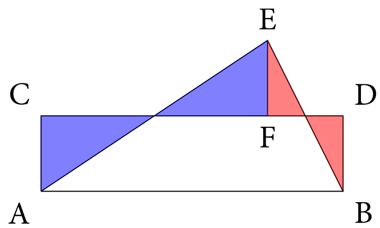


FIGURE 2. Lemma 1

All of the authors having established that for every triangle, there exists a rectangle of equal area, they all proceed with the following lemma, which requires Archimedes's axiom. While Jackson's and Hartshorne's proofs are identical, Lowry's is exceedingly complex and will not be reproduced here.

Lemma 2 ([WL14, Prob. II; Jac12, Theorem IV; Har00, Proposition 24.5])**.** *Given any rectangle $ABCD$ and any segment EF, there exists a rectangle $EFGH$ equivalent by dissection to $ABCD$. (Figure 3)*

Proof. We must fall into one of the following three cases:

(1) $\overline{AB} > 2\overline{EF}$;
(2) $2\overline{EF} \geq \overline{AB} \geq \overline{EF}$; or
(3) $\overline{EF} > \overline{AB}$.

It is sufficient to consider only the second case. Indeed, in the first case, by successively halving the base and consequently doubling the height a finite number of times, we obtain by the Archimedian axiom a rectangle satisfying the second case, which is equivalent by dissection to the original one by transitivity. Similarly, by successively doubling the base and consequently halving the height a finite number of times, we may also reduce the third case to the second case.

In the second case, let R lie on $\overline{AB}$ and S on $\overline{DC}$ such that $\overline{AR}$ and $\overline{SC}$ are congruent to $\overline{EF}$. Extend $\overline{AD}$ until it intersects with the ray $\vec{BS}$ in U, and draw the rectangle $ARVU$. Then by Euclid [Euc08a, I.29], the angles $\angle RBT$ and $\angle DSU$ are congruent, and so by Euclid [Euc08a, I.26], the triangles RBT and DSU are congruent. Similarly, since $\overline{SC} \cong \overline{AR} \cong \overline{UV}$, and the angles $\angle VUS$ and $\angle CSB$ are congruent by Euclid [Euc08a, I.29], the triangles TUV and BSC are congruent. So, subtracting congruent part from congruent triangles, we get that the quadrilaterals $USWV$ and $TWCB$ have equal content. Thus, the rectangles $ABCD$ and $ARVU$ have equal content, and $ARVU$ is the rectangle we seek. □

From here, Lowry, Jackson and Hartshorne all derive the theorem in the same manner, up to a choice of rectangle base in the proof:

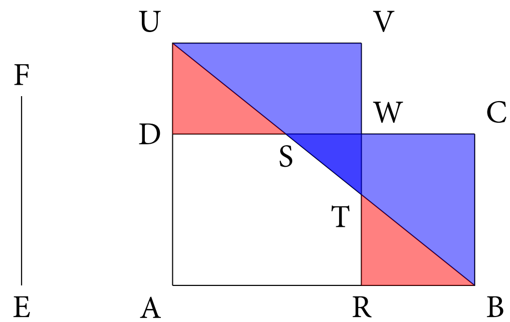


FIGURE 3. Lemma 2

Theorem 1 ([WL14, pp. 45f.; Jac12, Theorem V. Wallace's Theorem; Har00, Theorem 24.7])**.** *Two plane polygons with the same measure of area are congruent by dissection.*

Proof. Any polygon can be decomposed into triangles, and by Proposition 4 and Lemmas 1 and 2, each of these is respectively congruent by dissection to a rectangle with base 1. These rectangles may then be "stacked" to form a single rectangle with base 1. Thus, given any two polygons P and Q of the same area, there exist two rectangles with which they are respectively congruent by dissection, say, of dimensions 1 by a and 1 by b respectively. It follows that

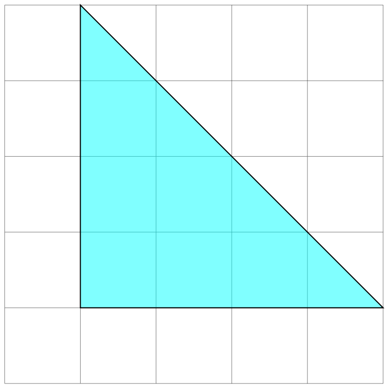
$$a = 1a = \mathrm{F}(P) = \mathrm{F}(Q) = 1b = b,$$

and so the two rectangles are congruent. Thus, the two polygons are congruent by dissection to the same rectangle, and so by Proposition 4 are themselves congruent by dissection. □

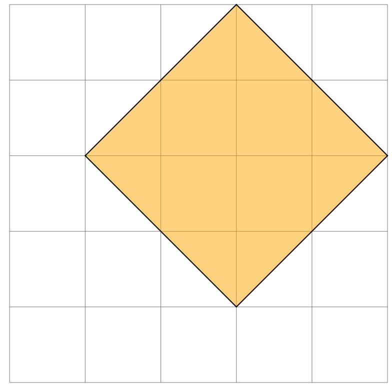
Corollary 2 (Wallace-Bolyai-Gerwien Theorem)**.** *Two plane polygons with the same measure of area have the same area.*

4. Example

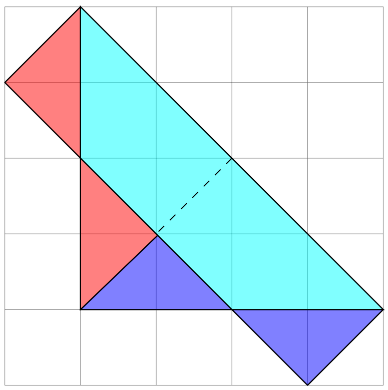
Beyond the examples of dissections given in Figures 2 and 3, we consider the task of showing that a right triangle and a square with the same measure of area have the same area. Our initial triangle and square are shown in Figures 4a and 4b. The triangle has measure of area 8 since the lengths of its base and its height are both four, while the square also has measure of area 8 since the length of its base is $2\sqrt{2}$. We begin by applying Lemma 1 to the triangle to transform it into a rectangle, as shown in Figure 4c. Cutting this rectangle in half lengthwise along the dotted line, we can rearrange it to form a rectangle, as shown in Figure 4d. Had the catheti of the initial triangle not been congruent, this rectangle would not have been a square, and we would have had to invoke Lemma 2 to get a square congruent to that of Figure 4b. The rearranging of triangles at the end would thus have been considerably messier. Fortunately, we chose our initial triangle to simplify presentation, and it so happens that this resulting rectangle is in fact a square congruent to the one in Figure 4b, since its sides are all easily shown to be of length $2\sqrt{2}$. Having rearranged the two half-rectangles to form the square of Figure 4b, we may now finish triangulating to get Figure 4e. Finally, we proceed in reverse order and rearrange the triangles of the triangulation to obtain anew our initial triangle in Figure 4f. We thus conclude that our initial triangle and square have the same area.



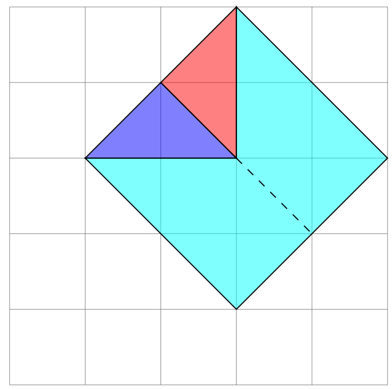
(A) An initial triangle



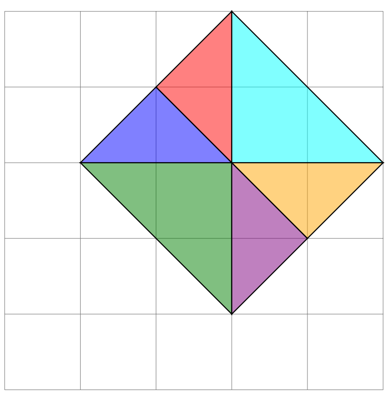
(B) A square with the same area



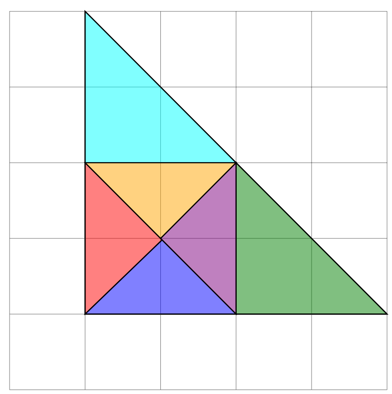
(C) Applying Lemma 1



(D) Rearranging the rectangles



(E) A final triangulation



(F) Rearranging back into the initial triangle

FIGURE 4. A square and a triangle with both the same measure of area and area

5. GENERALISATIONS

Several interesting questions naturally arise from the Wallace-Bolyai-Gerwien theorem. The theorem tells us that polygons of equal measure of area can be decomposed into respectively congruent triangles. What happens if we impose conditions on the triangles, for example, that they all have the same area? In 1965, Richman and Thomas asked:

Question 1 ([Hew+67, Question 5479])**.** Let N be an odd integer. Can a rectangle be dissected into N nonoverlapping triangles, all having the same area?

In 1970, Monsky used 2-adic valuations to provide an answer, proving what is now known as *Monsky's Theorem*:

Theorem 2 ([Mon70])**.** *A square can never be divided into an odd number of nonoverlapping triangles T_i, all of the same area.*

Hilbert also proposed a generalisation of the theorem. His third problem concerned its generalisation to three dimensions, using congruent tetrahedra instead of congruent triangles, and was the first of his 23 problems to be solved. He believed this generalisation was impossible, and so asked:

Question 2 ([Hil02b, Problem 3, p. 449])**.** Specify two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.

Dehn [Deh01], Hilbert's student, provided an impossibility proof using abstract algebra, and his proof introduced what is now known as the *Dehn invariant* for polyhedra. Sydler [Syd65] then succeeded in showing that an analogous decomposition of polyhedra is possible if and only if the two polyhedra have the same volume and Dehn invariant.

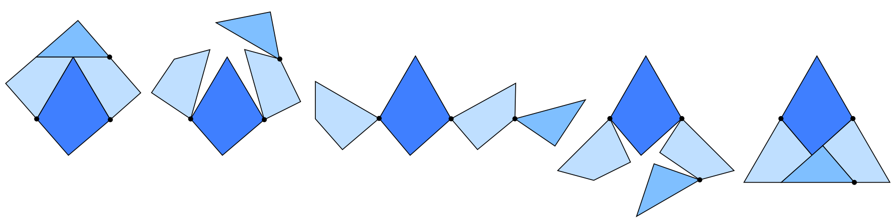


FIGURE 5. Dudeney's 1902 hinged dissection of a square into a triangle [Abb+12, Fig. 2]

The last generalisation is perhaps the most interesting. We define a *hinged dissection* to be "a chain of polygons hinged at vertices that can be folded in the plane continuously without self-intersection" [Abb+12]. For example, Figure 5 shows a hinge decomposition of a square into an equilateral triangle. Given two figures with the same area, one might wonder if it is always possible to find a hinge decomposition transforming one into the other. Not only does this happen to be the case, but Abbott et al. [Abb+12] proved the following much stronger result:

Theorem 3 ([Abb+12, Theorem 1])**.** *Any finite set of polygons of equal area have a common hinged dissection that can fold continuously without intersection between the polygons.*

That is to say, given a triangle, a pentagon, and a kite of equal areas, there exists a hinged dissection that forms the triangle in one configuration, the pentagon in another configuration, and the kite in yet another.

We conclude with two questions of our own:

Question 3. Does Monsky's Theorem generalise to cubes? That is, given a cube S, it has the same volume and Dehn invariant as itself, and so it can be decomposed into tetrahedra. Can this decomposition ever into an odd number of tetrahedra? For which positive integers N does there exist a decomposition into N tetrahedra?

Question 4. Bound from below the number of triangles required to show that two polygons have the same area.

6. Acknowledgements

The author gratefully acknowledges feedback from Melanie Ng, Ashna Raman, and Professor Gregory G. Smith.

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